J. ALGEBRAIC GEOMETRY
2 (1993) 385–387

SUBSETS OF $\mathbf{R}^n$ WHICH BECOME DENSE IN ANY COMPACT GROUP

FRANÇOIS ZIEGLER

Abstract

A polynomial curve or variety, contained in no proper affine subspace, becomes dense in the homomorphic image of $\mathbf{R}^n$ in any compact group.

1. Introduction

Let M be a subset of $\mathbf{R}^n$ and let E denote the smallest affine subspace containing M. We say that M is *Bohr dense in* E if $\beta(M)$ is dense in $\beta(E)$ whenever β is a continuous morphism of $\mathbf{R}^n$ into some compact group, T. We say that M is a *polynomial variety* if it is the image of a polynomial map $P : \mathbf{R}^m \to \mathbf{R}^n$. We propose to prove the following:

Theorem. *Let $M \subset \mathbf{R}^n$ be a polynomial variety. Then M is Bohr dense in its affine hull E.*

Thus a cone is Bohr dense in space, and so is a parabola in the plane; in contrast we observe that a hyperbola is not. Functions of the form $f \circ \beta$, with β as above and f a continuous function on T, are called *almost periodic*; so we may restate the theorem as:

Corollary. *Let $M \subset \mathbf{R}^n$ be a polynomial variety, contained in no proper affine subspace. Then every almost periodic function on $\mathbf{R}^n$ is determined by its restriction to M.*

2. Proof of the Theorem

Let all notation be as in the introduction. Translating everything so that M contains the origin, we may assume that E is a vector subspace of $\mathbf{R}^n$. Replacing $\mathbf{R}^n$ by this subspace and T by an abelian subgroup if necessary, we may also assume that $\overline{\beta(E)} = T$. We must then prove that $\overline{\beta(M)} = T$.

The idea is to show that the image of a "renormalized" Lebesgue measure under the given maps $\mathbf{R}^m \to E \to T$ coincides with Haar measure η

Received July 27, 1992.

on T. More precisely let λ denote Lebesgue measure on the cube $[0, 1]^m$ and let μ_a be the image of λ under the maps

$$[0, 1]^m \xrightarrow{a} \mathbf{R}^m \xrightarrow{P} E \xrightarrow{\beta} T,$$

where a stands for dilation by a factor $a \in \mathbf{R}$; then we shall prove that for every continuous function f on T one has

$$(*) \qquad \lim_{a\to\infty} \mu_a(f) = \eta(f).$$

The theorem follows: for if $f \geq 0$ vanishes on $\overline{\beta(M)}$ then so does the left-hand side and hence $\eta(f)$, which forces f to vanish everywhere.

Since linear combinations of characters are uniformly dense in the continuous functions on T, it is enough to prove $(*)$ when f is a character, in which case $\eta(f)$ is 1 or 0 according as $f \equiv 1$ or not. Since $(*)$ is clear when $f \equiv 1$, it remains to show that $\mu_a(f) = \lambda(f \circ \beta \circ P \circ a) \to 0$ as $a \to \infty$ whenever f is a nontrivial character of T. But then $f \circ \beta$ is a nontrivial character of E (because $\beta(E)$ is dense in T), so

$$(f \circ \beta \circ P)(x) = e^{i\langle \varphi, P(x)\rangle}$$

for a nonzero linear form φ on E. Moreover the polynomial $p(x) = \langle \varphi, P(x)\rangle$ is not constant on $\mathbf{R}^m$, for M is contained in no proper affine subspace of E. So matters are reduced to the following:

Lemma. *If p is a nonconstant polynomial on $\mathbf{R}^m$ then*

$$\lim_{a\to\infty} \int_{[0,1]^m} e^{ip(ax)}\, dx = 0.$$

Proof. Since p is not a constant, it has degree $k \geq 1$ in at least one of the variables, say $x_i = t$. Writing y for the remaining x_j and $p(x) = p_y(t)$, our integral I_a becomes the integral over $[0, 1]^{m-1}$ of

$$I_a(y) = \int_0^1 e^{ip_{ay}(at)}\, dt = \frac{1}{a}\int_0^a e^{ip_{ay}(t)}\, dt.$$

Now consider the coefficient $c(y)$ of t^k in $p_y(t)$: being a nonzero polynomial in y, it is nonzero for all y in a conull set $Y \subset [0, 1]^{m-1}$. Likewise $c(ay)$, for fixed $y \in Y$, is a nonzero polynomial in a and therefore nonzero for all a in a cofinite set $A_y \subset \mathbf{R}$. For fixed (y, a) in $Y \times A_y$ we conclude that the kth derivative $k!c(ay)$ of p_{ay} is bounded away from zero. By the generalized van der Corput lemma [1, p. 1258] this implies that

$$\left|\int_u^v e^{ip_{ay}(t)}\, dt\right| \leq \frac{2^{k+1}}{|k!c(ay)|^{1/k}} \qquad \forall u, v \in \mathbf{R}.$$

Taking $[u, v] = [0, a]$ and letting $a \to \infty$, it follows that $I_a(y) \to 0$ for all $y \in Y$, whence $I_a \to 0$ by dominated convergence. This completes the proof.

Acknowledgements

This work was supported by a grant of the Société Académique Vaudoise. I am also grateful to J. M. Souriau, J. Elhadad and B. Coupet for their help.

References

[1] E. M. Stein and S. Wainger, *Problems in harmonic analysis related to curvature*, Bull. Amer. Math. Soc. **84** (1978), 1239–1295.

Université d'Aix-Marseille II et CNRS-CPT Luminy, Case 907 F-13288 Marseille Cedex 09, France

E-mail address: ziegler@cptvax.in2p3.fr